Algebraic Aspects of Information Organisation

Ioan TOFAN and Aurelian Claudiu VOLF
Faculty of Mathematics, University "Al.I. Cuza", Iasi 6600, Romania
e-mail: tofan@uaic.ro ; volf@uaic.ro

ABSTRACT

In what follows, we approach the problem of information organization from the viewpoint of generalized structures (fuzzy structures and hyperstructures). The fuzzy quantitative information can be modelled by *fuzzy numbers,* while the fuzzy qualitative information has its counterpart in *hyperstructures*, in the sense that, for example, two (fuzzy) informations yield a *set* of possible consequences. The significance of information appears most clearly in *structures*; this induces the necessity of studying the *fuzzy algebraic structures* (fuzzy groups, rings, ideals, subfields and so on) as a means towards the better understanding and processing of information. This report presents some recent results and methods in the rapidly growing fields of fuzzy algebraic structures and hyperstructures and some connections between them. Some results on fuzzy groups, fuzzy rings and fuzzy subfields are given. Likewise, the consideration of diverse *sets of fuzzy numbers* and, more notably, of the *structures* that these sets can be endowed with is of utmost importance. In this direction, the *operations* with fuzzy numbers play a major role and a number of questions regarding these operations are still open. A sample of the different notions of fuzzy number and of the operations with fuzzy numbers and their properties is given in this report. The similarity relations (fuzzy generalizations of equivalence relations) are in direct connection with shape (pattern) recognition. Diverse types of similarity classes and partitions are studied. Several notions of *f*-hypergroup, which combine fuzzy structures and hyperstructures, are presented and studied. Some results that put forward a two-way connection between *L*-fuzzy structures and hyperstructures are given.

1. Introduction

In what follows, we deal with the problem of information organization from the viewpoint of generalized structures (fuzzy structures and hyperstructures).

Generally speaking, one can accept the fact that “to solve a problem (not necessarily of a mathematical nature)” means “to determine a set” (the set of the solutions), based upon the problem data (that is, upon a set of informations). But, to determine a set means to give a characteristic property, in other words, to obtain an information. In this context, a classification of properties (informations) may be useful. One can distinguish between *qualitative properties* (corresponding to the *linguistic level* of information) and *quantitative properties* (corresponding to the *numerical level* of information). In most cases, the information is not crisp, precise, but vague and imprecise, "fuzzy". The fuzzy quantitative information can be modelled by *fuzzy numbers,* while the fuzzy qualitative information has its counterpart in *hyperstructures*, in the sense that, for example, two (fuzzy) informations yield a *set* of possible consequences.

The significance of information appears most clearly in structures; this induces the necessity of studying the *fuzzy algebraic structures* (fuzzy groups, rings, ideals, subfields and so on) as a means towards the better understanding and processing of information. The theory of algebraic hyperstructures has surprising connections with the fuzzy structures, which can be interpreted as connections between the two types of information described above. The similarity relations (fuzzy generalizations of equivalence relations) are in direct connection with shape (pattern) recognition.

This report presents some recent results and methods in the rapidly growing fields of fuzzy algebraic structures and hyperstructures and some connections between them. Some results on fuzzy groups, fuzzy rings and fuzzy subfields are given. Likewise, the consideration of diverse *sets of fuzzy numbers* and, more notably, of the *structures* that these sets can be endowed with is of utmost importance. In this direction, the *operations* with fuzzy numbers play a major role and a number of questions regarding these operations are still open. A sample of the different notions of fuzzy number and of the operations with fuzzy numbers and their properties is given in this report. Diverse types of similarity classes and partitions are studied. Several notions of *f*-hypergroup, which combine fuzzy structures and hyperstructures, are presented and studied. Some results that put forward a two-way connection between *L*-fuzzy structures and hyperstructures are given.

2. Preliminaries

2.1 Fuzzy sets

The theory of fuzzy sets extends the area of applicability of mathematics, by building the instruments and the framework for the management of the imprecision inherent to the human language and thinking. The starting point is generalizing the notion of subset of a set. It is well-known that a subset A' of the set A is perfectly determined by its *characteristic function* $c_{A'} : A' \ ?\ \ \{0,1\},\ c_{A'}(x) = \begin{cases} 1, \text{if } x \in A' \\ 0, \text{otherwise} \end{cases}.$

One generalizes the notion of "belonging to" the subset *A'* by introducing a gradual transition from "does not belong to" to "belongs to" (L. Zadeh, 1965). L. Zadeh succeded in imposing the theory of fuzzy sets, by exhibiting applications of the theory. The idea of rejecting the principle "tertium non datur" is directly connected to the generalization above. It goes back to Aristotle and appears in the modal logic (Mac Coll, 1897) or multivalued logics. The generalization of the concept of "characteristic function" was given by H. Weyl (1940) and appears again in a new interpretation in papers by A. Kaplan & H. F. Schott and K. Menger.

1.1 DEFINITION. Let U be a nonempty set. A pair (U, m), where $m : U\ ?\ \ [0,1]$ is a mapping, is called a *fuzzy set*. If $x \in U$, $m(x)$ is understood as the "degree to which x belongs to the fuzzy set determined by m". We shall also call $m : U\ ?\ \ [0,1]$ a *fuzzy subset* of U and denote $\mathrm{F}(U) := [0,1]^U = \{m \mid m : U\ ?\ \ [0,1]\}$ the set of fuzzy subsets of U.

It is sometimes useful to replace the interval [0,1] (which is a lattice with respect to the usual order relation) with a lattice $(L, \wedge, \vee)$. Thus, a pair (U, m), where $m : U\ ?\ \ L$, is called an *L-fuzzy set* or *L-fuzzy subset* of U. Many definitions and results on fuzzy sets can be transferred to L-fuzzy sets, provided some conditions on L are imposed.

1.2 DEFINITION. Let (U, m) be a fuzzy set and $a \in [0,1]$. The set

$$_{m}U_{a} := \{x \in U \mid m(x) \geq a\}$$

(also denoted m_a) is called the *a-level set* of (U, m). Let supp$m := \{x \in U \mid m(x) \neq 0\}$.

1.3 PROPOSITION. *Let* $(U_a)_{a \in [0,1]} \subseteq P(U)$ *be a family of subsets of U. Then* $(U_a)_{a \in [0,1]}$ *is the family of level sets of a fuzzy subset* $m : U\ ?\ [0,1]$ *if and only if it satisfies the conditions:*

a) $U_0 = U$.

b) $\forall a, b \in [0,1]$, $a \leq b$ *implies* $U_b \subseteq U_a$.

c) For any increasing sequence $(a_i)_{i \geq 0}$, $a_i \in [0,1]$, $\forall i \in \mathbb{N}$, *having limit* a, *we have* $U_a = \bigcap_{i \geq 0} U_{a_i}$.

A fuzzy set is completely determined by the family of its level sets.:

1.4 PROPOSITION. *Let X be a set and let* m *a fuzzy subset of X. Then*

$$m(x) = \sup\{k \in [0,1] \mid x \in {}_{m}X_{k}\}.$$

1.5 DEFINITION. i) $m_{\emptyset} \in F(U)$ given by $m_{\emptyset}(x) = 0$, $\forall x \in U$, is called the *empty fuzzy subset* of U.

ii) If $m, t \in F(U)$, the *inclusion* $m \subseteq t$ is defined by $m(x) \leq t(x)$, $\forall x \in U$.

iii) If $m, t \in F(U)$, define $m?\ t$ (the *union* of the fuzzy subsets m and t) by $m?\ t : U\ ?\ [0,1]$, $(m?\ t)(x) = \max\{m(x), t(x)\}$. The *intersection* is defined by $m \cap t : U\ ?\ [0,1]$, $(m \cap t)(x) = \min\{m(x), t(x)\}$. These definitions extend to families of fuzzy subsets: if $\{m_i\}_{i \in I} \subseteq F(U)$, then we set:

$$\bigcap_{i \in I} \mu_i : U \to [0,1], \left(\bigcap_{i \in I} \mu_i\right)(x) = \inf_{i \in I}\{\mu_i(x)\};\ \bigcup_{i \in I} \mu_i : U \to [0,1], \left(\bigcup_{i \in I} \mu_i\right)(x) = \sup_{i \in I}\{\mu_i(x)\}.$$

v) For $m \in F(U)$, the fuzzy subset $m' \in F(U)$ given by $m'(x) = 1 - m(x)$, $\forall x \in U$, is called the *complement* of m.

1.6 REMARK. $(F(U), \cap, ?, ')$ is a de Morgan algebra (as opposed to $(P(U), \cap, \cup, \neg)$ which is a Boole algebra). Note that $\{0, 1\}$ has a Boole algebra structure (with respect to min, max, $x' = 1 - x$), while $[0, 1]$ with the same operations is just a de Morgan algebra. On $F(U)$ the following operations can be defined:

"$+$" by $m + t : U\ ?\ [0,1]$, $(m + t)(x) = m(x) + t(x) - m(x)t(x)$, $\forall x \in U$.

"$\cdot$" by $m \cdot t : U\ ?\ [0,1]$, $(m \cdot t)(x) = m(x) \cdot t(x)$, $\forall x \in U$.

"?" by $m\ ?\ t : U\ ?\ [0,1]$, $(m\ ?\ t)(x) = \min\{1, m(x) + t(x)\}$, $\forall x \in U$;

"?" by $m\ ?\ t : U\ ?\ [0,1]$, $(m\ ?\ t)(x) = \max\{0, m(x) + t(x) - 1\}$, $\forall x \in U$

2.2 Hyperstructures

The concept of hypergroup was introduced in 1934 by F. Marty as a natural generalization of the notion of group. Many applications in geometry, combinatorics, group theory, automata theory etc. have turned hypergroup theory and subsequently hyperstructure theory into a relevant domain of modern algebra.

2.1 DEFINITION. Let H be a nonempty set. Let $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\} = \{A \mid A \subseteq H, A \neq \emptyset\}$. A *hyperoperation* "$*$" on H is mapping $* : H \times H$? $\mathcal{P}^*(H)$. For any $a \in H$ and $B \subseteq H$, $B \neq \emptyset$, we denote by $a * B = \bigcup\limits_{b \in B} a * b$. Similarly one defines $B * a$. If $A, B \in \mathcal{P}^*(H)$, let $A * B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a * b$.

A nonempty set endowed with a hyperoperation "$*$" on H is called a *hypergroupoid*. If, $\forall a, b, c \in H$, we have $a * (b * c) = (a * b) * c$ (*associativity*), then H is called a *semihypergroup*. If a semihypergroup $(H, *)$ satisfies $a * H = H * a = H$, $\forall a \in H$ (*reproducibility*) then H is called a *hypergroup*. A hypergroup is called *commutative* if, $\forall a, b \in H$, $a * b = b * a$.

2.2. REMARK. A hyperoperation $*$ defined on a set H induces two hyperoperations "/" and "\". For every $x, y \in H$, define:

$$x / y = \{a \in H \mid x \in a * y\}, \qquad x \backslash y = \{b \in H \mid x \in y * b\}.$$

If "$*$" is commutative, then $x / y = x \backslash y$, $\forall x, y \in H$. Also, the reproducibility axiom is equivalent to the condition: $\forall x, y \in H$, $x / y \neq \emptyset$ and $x \backslash y \neq \emptyset$.

2.3. DEFINITION. A commutative hypergroup $(H, *)$ is called a *join space* if, $\forall a, b, c, d \in H$, $a/b \cap c/d \neq \emptyset$ implies $a * d \cap b * c \neq \emptyset$.

3. Fuzzy algebraic structures

3.1 Fuzzy subgroups

1.1 DEFINITION. Let $(G, \cdot, e)$ be a group and let $m : G \to [0, 1]$ be a fuzzy subset of G. We say that m is a *fuzzy subgroup* of G if :

i) $m(xy) \geq \min\{m(x), m(y)\}$, $\forall x, y \in G$;

ii) $m(x^{-1}) \geq m(x)$, $\forall x \in G$.

If moreover $m(xy) = m(yx)$, $\forall x, y \in G$, then m is called a *normal fuzzy subgroup* of G.

1.2 REMARK. If m is a fuzzy subgroup of G, then $m(x^{-1}) = m(x) \leq m(e)$, $\forall x \in G$. Moreover, m is a normal fuzzy subgroup if and only if $m(y^{-1}xy) \geq m(x)$, $\forall x, y \in G$.

The next characterization is typical for all "fuzzy substructures".

1.3 PROPOSITION. *A fuzzy set* $m : G$? $[0, 1]$ *is a (normal) fuzzy subgroup of G if and only if the level subsets* ${}_mG_a$ *are (normal) subgroups of G for all* $a \in \mathrm{Im}m$.

1.4 DEFINITION. We say the fuzzy set (F, m) satisfies the *sup property* if, for every nonempty subset A of $\mathrm{Im}m$, there exists $x \in \{y \in F \mid m(y) \in A\}$ such that $m(x) = \sup A$. In other words, m has the sup property if and only if any nonempty subset A of $\mathrm{Im}m$ has a greatest element.

1.5 PROPOSITION. *Let* $(G, \cdot, e)$, $(H, \cdot, e')$ *be groups,* $f : G$? H *group homomorphism and* m, h *fuzzy subgroups of G, respectively H. Then* $f^{-1}(h)$ *is a fuzzy subgroup of G. If* (G, m) *has the sup property, then* $f(m)$ *is a fuzzy subgroup of H.*

3.2 Fuzzy ideals

2.1 DEFINITION. Let $(R, +, \cdot)$ be a unitary commutative ring.

i) A fuzzy subset $s : R \to I$ is called a *fuzzy subring* of R if, $\forall x, y \in R$:

$$m(x - y) \geq \min\{m(x), m(y)\}; \quad m(xy) \geq \min\{m(x), m(y)\}.$$

ii) A fuzzy subset $s : R \to I$ is called a *fuzzy ideal* of R if, $\forall x, y \in R$:

$$m(x - y) \geq \min\{m(x), m(y)\}; \quad m(xy) \geq \max\{m(x), m(y)\}.$$

2.2 PROPOSITION. *Let* $m : R \to [0, 1]$ *be a fuzzy ideal of* R. *Then*:

i) $m(1) = m(x) = m(-x) = m(0)$, $\forall x \in R$;

ii) $m(x - y) = m(0) \Rightarrow m(x) = m(y)$, $\forall x, y \in R$;

iii) $m(x) < m(y)$, $\forall y \in R \Rightarrow m(x - y) = m(x) = m(y - x)$.

2.3 PROPOSITION. *A fuzzy subset* $m : R \to [0, 1]$ *is a fuzzy subring* (*ideal*) *of* R *if and only if all level subsets* ${}_mR_a$, $a \in \mathrm{Im}m$, *are subrings* (*ideals*) *of* R.

2.4 REMARK. The intersection of a family of fuzzy ideals of R is a fuzzy ideal of R. This leads to the notion of *fuzzy ideal generated* by a fuzzy subset s of R, namely the intersection of all fuzzy ideals that include s, denoted $< s >$. We have: $< s > : R \to [0, 1]$ is given by $< s > (x) = \sup\{a \in [0,1] \mid x \in <{}_mR_a >\}$.

2.5 PROPOSITION. *The union of a totally ordered* (*with respect to the relation* $m \leq h \Leftrightarrow m(x) \leq h(x)$, $\forall x \in R$) *family of fuzzy ideals of* R *is a fuzzy ideal of* R.

2.6 DEFINITION. Let m, q be fuzzy ideals of R. The *product of* m and q is:

$$m \cdot q : R \to [0, 1], \ (m \cdot q)(x) = \sup_{\substack{x = \sum y_i z_i \\ i < \infty}} \{ \min_i \{\min\{m(y_i), q(z_i)\}\}\}, \ \forall x \in R.$$

The *sum* of m and q is:

$$m + q : R \to [0, 1], \ (m + q)(x) = \sup\{\min\{m(y), q(z)\} \mid y, z \in R, y + z = x\}, \ \forall x \in R.$$

2.7 REMARK. In general, for m, q fuzzy subsets of a set S endowed with a binary operation "$\cdot$", one defines the *product* $mq : S \to [0, 1]$,

$$(mq)(x) = \begin{cases} \sup\limits_{yz=x}\{\min\{m(y), q(z)\}\}, & \text{if there exist } y, z \in S \text{ such that } x = yz, \\ 0, & \text{otherwise} \end{cases}$$

For any m, q fuzzy ideals of R, we have $m \cdot q = <mq>$.

2.8 PROPOSITION. *Let* $f : R \to R'$ *be a surjective ring homomorphism and* m *a fuzzy ideal of* R, m' *a fuzzy ideal of* R'. *Then*:

i) $f(m)$ *is a fuzzy ideal of* R';

ii) $f^{-1}(m')$ *is a fuzzy ideal of* R.

2.9 DEFINITION. A nonconstant fuzzy ideal m ($|\mathrm{Im}\ m| > 1$) of a ring R is called a *fuzzy prime ideal* if, for any fuzzy ideals s, q of R, $sq \subseteq m \Rightarrow s \subseteq m$ or $q \subseteq m$.

3.3 Fuzzy rings of quotients

The study of the fuzzy prime ideals of a ring leads naturally to the question of the existence of a "fuzzy localisation" device, that is, to the problem of the construction of a fuzzy ring of quotients. Let R be unitary commutative ring. R^* denotes the set of the invertible elements of R.

3.1 DEFINITION. A fuzzy subset $s : R \to [0,1]$ is called a *fuzzy multiplicative subset* (FMS for short) if:

i) $s(xy) \geq \min(s(x), s(y)), \forall x, y \in R.$

ii) $s(0) = \min\{s(x) : x \in R\}$;

iii) $s(1) = \max\{s(x) : x \in R\}$.

3.2 PROPOSITION. *The fuzzy subset s of the ring R is a FMS if and only if every level subset $s_t = \{x \in R : s(x) \geq t\}$, $t > s(0)$, is a multiplicative system (in the classical sense).*

Recall that a multiplicative subset S of R is *saturated* if $xy \in S$ implies $x, y \in S$.

3.3 DEFINITION. A FMS s of a ring R is called *saturated* if, for any $x, y \in R$,

$$s(xy) = \min(s(x), s(y)).$$

3.4 PROPOSITION. *The fuzzy subset s of the ring R is a saturated FMS if and only if every level subset s_t is a saturated multiplicative system, $\forall\, t > s(0)$.*

3.5 PROPOSITION. *If m is a fuzzy prime ideal of a ring R, then $1-m$ is a saturated FMS.*

3.6 PROPOSITION. *Let s be a FMS of the ring R. Then the fuzzy subset $\bar{s}$, defined by*

$$\bar{s}(x) = \sup\{s(xy) : y \in R\}$$

is a saturated FMS, with $s \leq \bar{s}$. Moreover, if t is a saturated FMS with $s \leq t$, then $\bar{s} \leq t$.

This result entitles us to call $\bar{s}$ above the *saturate* of s.

Let s be a FMS of the ring R and $m = s(0)$. For every $t > m$, we may construct the classical ring of fractions $s_t^{-1}R = S_t$ with respect to the multiplicative subset s_t. Let j_t denote the canonical ring homomorphism $R \to S_t$. If $s < t$, since $s_t \subseteq s_s$, the universality property of the ring of fractions yields the existence of a unique ring homomorphism $j_{ts} : S_t \to S_s$ such that $j_{ts} \circ j_t = j_s$. The system of rings and homomorphisms (S_t, j_{ts}), t, $s \in\,]m, 1]$ is an inductive system (if $]m, 1]$ is endowed with the reverse of the usual order). Let $s^{-1}R$ denote the inductive limit of this system and let j be the canonical ring homomorphism $R \to s^{-1}R$ (the inductive limit of the j_t, $t > m$).

It is natural to call $s^{-1}R$ the *ring of quotients* relative to the FMS s.

3.7 PROPOSITION. *With the notations above, j has the following universality property: for every $t > s(0)$, $j(s_t) \subseteq (s^{-1}R)^*$; if T is a ring and $y : R \to T$ is a ring homomorphism such that for every $t > s(0)$, $j(s_t) \subseteq T^*$, then it exists a unique ring homomorphism $f : s^{-1}R \to T$ such that $f \circ j = y$.*

3.8 PROPOSITION. *There is a canonical isomorphism $y : s^{-1}R \to \bar{s}^{-1}R$. If $\bar{j} : R \to \bar{s}^{-1}R$ denotes the canonical homomorphism, then $\bar{j} = y \circ j$.*

By applying Zorn's Lemma to the set **P**, one proves**:**

3.9 PROPOSITION. *If* s *is a FMS in R and* m *is a fuzzy ideal such that* $m \cap s = \varnothing$, *then the set* $\mathbf{P} = \{h : h$ *is a fuzzy ideal of R,* $h \cap s = \varnothing$, $m \subseteq h\}$ *has maximal elements and any such element is a fuzzy prime ideal. Thus it exists a fuzzy prime ideal* p *such that* $p \cap s = \varnothing$.

3.10 PROPOSITION. *Let* p *be a fuzzy prime in R and denote by* R_p *the ring* $(1 - p)^{-1}R$. *Then* R_p *is a local ring.*

3.4 Fuzzy intermediate fields

Let F/K be a field extension and let $\mathrm{I}(F/K) = \{L|\ L$ subfield of F, $K \subseteq L\}$ be the lattice of its *intermediate fields* (we also called them *subextensions* of F/K). If F/K is a field extension and $c \in F$ is algebraic over K, then we denote by $\mathrm{Irr}(c, K) \in K[X]$ the minimal polynomial of c over K.

4.1. DEFINITION. Let F/K be an extension of fields and $m : F\ ?\ [0,1]$ a fuzzy subset of F. We call m a *fuzzy intermediate field* of F/K if, $\forall x, y \in F$:

$m(x - y) \geq \min\{m(x), m(y)\}$;

$m(xy^{-1}) \geq \min\{m(x), m(y)\}$ if $y \neq 0$;

$m(x) \leq m(k)$, $\forall k \in K$.

Let $\mathrm{FI}(F/K)$ denote the set of all fuzzy intermediate fields of F/K.

If $m \in \mathrm{FI}(F/K)$, then m is a constant on K.

For any fuzzy subset $m : F\ ?\ [0,1]$ and $s \in [0,1]$, define the *level set* $m_s := \{x \in F\ |m(x) \geq s\}$.

It is well known that *a fuzzy subset* $m : F\ ?\ [0,1]$ *is a fuzzy intermediate field if and only if,* $\forall s \in \mathrm{Im}\,m$, *the level set* m_s *is an intermediate field of* F/K.

4.2. THEOREM. *Let F/K be a field extension. Then every fuzzy intermediate field of F/K has the sup property iff there are no infinite strictly decreasing sequences of intermediate fields of F/K.*

4.3. REMARK. This result can be applied, mutatis mutandis, to *any algebraic structure* for which is defined a notion of "fuzzy substructure". For instance, let $(G, \cdot)$ be a group and 1 is its neutral element. By replacing in Theorem 4.2 "intermediate field" with "subgroup" and K (the base field) with the trivial subgroup $\{1\}$, one obtains the following fact:

4.4. PROPOSITION. *Let G be a group. Then every fuzzy subgroup of G has the sup property if and only if there are no infinite strictly decreasing sequences of subgroups of G.*

Similarly, in the case of *fuzzy ideals*, we have:

4.5. PROPOSITION. *Let R be a unitary commutative ring. Then every fuzzy ideal of R has the sup property if and only if R is Artinian* (*there are no infinite strictly decreasing sequences of ideals of R*).

4.6. DEFINITION. [2] Let F/K be an extension of fields and $m \in \mathrm{FI}(F/K)$. Then m is called a *fuzzy chain subfield* of F/K if $\forall x, y \in F$, $m(x) = m(y) \Leftrightarrow K(x) = K(y)$.

Here is a fuzzy characterization of the fact that $\mathrm{I}(F/K)$ is a chain.

4.7. THEOREM. [[2], Th. 3.3]. *The intermediate fields of F/K are chained if and only if F/K has a fuzzy chain subfield.*

4.8. THEOREM. *Let F/K be an extension such that the intermediate fields of F/K are chained. Then:*

a) F/K is algebraic.

b) Any intermediate field L of F/K with $L \neq F$ is a finite simple extension of K.

c) $(\mathcal{I}(F/K), \subseteq)$ satisfies the descending chain condition (there is no strictly decreasing sequence of intermediate fields of F/K). Thus, $(\mathcal{I}(F/K), \subseteq)$ is well ordered.

4.9. COROLLARY. *Let F/K be a field extension.*

a) Assume that any proper intermediate field of F/K is a finite extension of K. Then every $m \in \mathcal{FI}(F/K)$ has the sup property.

b) If the intermediate fields of F/K are chained, then every $m \in \mathcal{FI}(F/K)$ has the sup property.

c) If every $m \in \mathcal{FI}(F/K)$ has the sup property, then F/K is algebraic.

4. Applications and connections

4.1 Fuzzy numbers

1.1. DEFINITION. Let $(G, \cdot)$ be a set endowed with a binary operation "$\cdot$" (usually a group). Let m, $q \in \mathcal{F}(G)$. We use the definition 3.2.7 for $m * q \in \mathcal{F}(G)$,

$$(m*q)(x)=\begin{cases}\sup\limits_{yz=x}\{\min\{m(y),q(z)\}\}, \text{if there exist } y, z \in S \text{ such that } x = yz,\\ 0, \quad \text{otherwise}\end{cases}$$

Thus, "$*$" is a binary operation on $\mathcal{F}(G)$.

If G is a group and e is its neutral element, we denote $c_{\{e\}}$ by e. For any $m \in \mathcal{F}(G)$, let $\tilde{m}$: $G \to [0, 1]$, $\tilde{m}(x) = m(x^{-1})$, $\forall x \in G$.

1.2. PROPOSITION. *Let G be a group.*

i) *The operation* "$*$" *on* $\mathcal{F}(G)$ *is associative*;

ii) *If G is commutative, then* "$*$" *is commutative*.

iii) $\forall m \in \mathcal{F}(G)$, $m * e = e * m = m$;

iv) $e \subseteq m * \tilde{m}$, $e \subseteq \tilde{m} * m$.

1.3. PROPOSITION. *Let m, t, $n \in \mathcal{F}(G)$. Then:*

i) $m \subseteq t \Rightarrow m * n \subseteq t * n$, $n * m \subseteq n * t$;

ii) $m * (t \cup n) = (m * t) \cup (m * n)$; $(t \cup n) * m = (t * m) \cup (n * m)$;

iii) $m * (t \cap n) = (m * t) \cap (m * n)$; $(t \cap n) * m = (t * m) \cap (n * m)$.

1.4. DEFINITION. *A fuzzy number* is a mapping $m : \mathbb{R} \to [0, 1]$ (where $\mathbb{R}$ is the field of real numbers) such that there exists $x_m \in \mathbb{R}$ with $m(x_m) = 1$, the set $\{x \mid m(x) \neq 0\}$ is bounded and the level sets $_m\mathbb{R}_a$ are closed intervals ($a \in [0,1]$).

For any $r \in \mathbb{R}$, the mappings $\tilde{r} : \mathbb{R} \to [0, 1]$, $\tilde{r}(x) = \begin{cases}1, & x = r\\ 0, & \text{otherwise}\end{cases}$, are called *degenerate fuzzy numbers.*

One usually takes the fuzzy numbers of the following type:

$$m(x) = \begin{cases} 0, & x < a \quad ; \\ p_1(x), & x \in [a, b); \\ 1, & x \in [b, c]; \\ p_2(x), & x \in (c, d]; \\ 0, & x > d. \end{cases} \tag{1}$$

where $a \le b \le c \le d$ are reals, and $p_1,\ p_2$: $\mathbb{R} \to \mathbb{R}$ satisfy the conditions that turn m in a fuzzy number as in the definition.

For $p_1(x) = \dfrac{x-a}{b-a}$, $p_2(x) = \dfrac{d-x}{d-c}$ one gets *trapezoidal fuzzy numbers*. If $b = c$, *triangular fuzzy numbers* are obtained. A trapezoidal fuzzy number as above is denoted by $A = (a, b, c, d)$, respectively $A = (a, b, d)$ for triangular fuzzy numbers.

The operations with fuzzy numbers m, h are defined as in the case of $F(G)$ above:

$$m * h\colon \mathbb{R} \to [0, 1],\ (m * h)(z) = \sup_{x \circ y = z} \{\min\{m(x), h(y)\}\},$$

By replacing "?" with "+", "·", "–", ":", one obtains the operations "? ", respectively "? ", "? ", "? ".

We use the following notations:

- $\mathcal{R}$ is the set of nondegenerate fuzzy numbers;
- $\mathcal{R}_+ = \{m \in \mathcal{R} \mid m(x) > 0 \Rightarrow x > 0\}$, $\mathcal{R}_- = \{m \in \mathcal{R} \mid m(x) > 0 \Rightarrow x < 0\}$,
- $\mathcal{R}^* = \mathcal{R}_+ ?\ \mathcal{R}_-$;

1.5. Remark. For any $m, h \in \mathcal{R}$ and $r \in \mathbb{R}$, we have, $\forall x \in \mathbb{R}$:

$(\tilde{r}\ ?\ m)(x) = m(r - x)$; $\qquad (m\ ?\ h)(x) = \sup\limits_{y \in \mathbb{R}} \{\min\{m(y), h(x - y)\}\}$;

$$(\tilde{r}\ ?\ m)(x) = \begin{cases} m\left(\dfrac{x}{r}\right), & r \neq 0 \\ \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}, & r = 0 \end{cases}; \qquad (m\ ?\ h)(x) = \begin{cases} \sup\limits_{y \neq 0}\left\{\min\left\{\mu(y), \mu\left(\dfrac{x}{y}\right)\right\}\right\}, & x \neq 0 \\ \max\{\mu(0), h(0)\}, & x = 0 \end{cases},$$

1.6 Remark. Fuzzy numbers can be characterized by a family of intervals (*intervals of confidence*). Let $m \in \mathcal{R}$, $a \in [0, 1]$. Define $m_a = [\underline{x}_a, \overline{x}_a]$, where $\underline{x}_a = \inf\{x \mid m(x) = a\}$, $\overline{x}_a = \sup\{x \mid m(x) = a\}$. If m is of the type (1), we get:

$$[\underline{x}_a, \overline{x}_a] = \begin{cases} [p_1^{-1}(\{a\}), p_2^{-1}(\{a\})], & a \neq 1 \\ [b, c] & a = 1 \end{cases}.$$

The conditions p_1 strictly increasing and p_2 strictly decreasing determine the fuzzy number if the confidence intervals are given. For the numbers of the type $\tilde{r}$ ($r \in \mathbb{R}$) the use of confidence intervals is superfluous. In this context the operations with fuzzy numbers can be defined as follows: $\forall m, h \in \mathcal{R}$, with $m_\alpha = [\underline{x}_a, \overline{x}_a]$, $h_a = [\underline{y}_a, \overline{y}_a]$, we define:

$$(m \oplus h)_a = [\underline{x}_\alpha + \underline{y}_a,\ \overline{x}_a + \overline{y}_a];\ (m\ ?\ h)_a = [\underline{x}_a - \overline{y}_a,\ \overline{x}_a - \underline{y}_a];$$

$$(m ?\ h)_a = [\min\{\underline{x}_a \underline{y}_a, \underline{x}_a \overline{y}_a, \overline{x}_a \underline{y}_a, \overline{x}_a\ \overline{y}_a\}, \max\{\underline{x}_a \underline{y}_a, \underline{x}_a \overline{y}_a, \overline{x}_a \underline{y}_a, \overline{x}_a\ \overline{y}_a\}];$$

$$(m ?\ h)_a = \left[\min\left\{\frac{\underline{x}_a}{\underline{y}_a}, \frac{\overline{x}_a}{\underline{y}_a}, \frac{\underline{x}_a}{\overline{y}_a}, \frac{\overline{x}_a}{\overline{y}_a}\right\}, \max\left\{\frac{\underline{x}_a}{\underline{y}_a}, \frac{\overline{x}_a}{\underline{y}_a}, \frac{\underline{x}_a}{\overline{y}_a}, \frac{\overline{x}_a}{\overline{y}_a}\right\}\right], \text{ if } 0 \notin h_a, (a \in (0,1)).$$

1.7 REMARK. For trapezoidal or triangular fuzzy numbers, $A = (a, b, c, d)$, respectively $A = (a, b, c)$, the confidence intervals are $A_a = [(b - a)a + a, (c - d)a + d]$, respectively $A_a = [(b - a)a + a, (b - c)a + c]$, $a \in [0, 1]$.

In these cases, $A_0 = [a, c]$, respectively $A_1 = [a, b]$.

Since A_0 and A_1 determine completely the triangular fuzzy number m, sometimes it is taken the following definition (for $A = (a_1, b_1, c_1)$, $B = (a_2, b_2, c_2)$):

$$A \oplus B = (a_1 + a_2, b_1 + b_2, c_1 + c_2); \quad A ?\ B = (a_1a_2, b_1b_2, c_1c_2), \text{ for } a_1, a_2 \geq 0;$$

$$A ?\ B = (a_1 - a_2, b_1 - b_2, c_1 - c_2); \quad A ?\ B = \left(\frac{a_1}{c_2}, \frac{b_1}{b_2}, \frac{c_1}{a_2}\right), \text{ for } a_1, a_2 > 0.$$

If A is a triangular fuzzy number, $A = (a, b, c)$, we denote also $-A = (-a, -b, -c)$ and $A^{-1} = (c^{-1}, b^{-1}, a^{-1})$ if $a > 0$.

For any $a \in \mathbb{R}$, let $0_a = (-a, 0, a)$ and for any $a \geq 1$, let $1_a = (a^{-1}, 1, a)$. We have $0_a \oplus 0_b = 0_{a+b}$, $1_a ?\ 1_b = 1_{ab}$.

1.8 DEFINITION. We define on the set of triangular fuzzy numbers $\mathbb{R}_t$ the following relations : $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ are $\oplus$-*equivalent* (and write $A_1 \sim_\oplus A_2$) if there exist 0_a, 0_b such that $A_1 \oplus 0_a = A_2 \oplus 0_b$; we say that $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ are ? -*equivalent* (we write $A_1 \sim_?\ A_2$) if there exist 1_a, 1_b such that $A_1 ?\ 1_a = A_2 ?\ 1_b$.

It is easy to see that $0_a \sim_\oplus 0_b$ for every $a, b \in \mathbb{R}$ and $1_a \sim_?\ 1_b$ for every $a, b \geq 1$.

1.9 PROPOSITION. *The relations* $\sim_\oplus, \sim_?$ *are equivalence relations.*

Let $\mathbb{R}_\oplus = \mathbb{R}_t/\sim_\oplus$ and for every $A \in \mathbb{R}_t$ denote $\overline{A} \in \mathbb{R}_\oplus$ the equivalence class of A. For $\overline{A}$, $\overline{B} \in \mathbb{R}_\oplus$, we define $\overline{A}\ [+]\ \overline{B} = \overline{A[+]B}$.

1.10 PROPOSITION. *The operation* [+] *is well defined and* $(\mathbb{R}_\oplus, [+])$ *is an abelian group,* $\overline{0_a}$ *being its neutral element* $(\forall a \in \mathbb{R})$ *;* $\overline{-A}$ *is the symmetrical element of* $\overline{A}$.

1.11 REMARK. A similar result can be obtained for $\mathbb{R}_? = {}_+\mathbb{R}_t/\sim_?$, where ${}_+\mathbb{R}_t$ is the set of triangular fuzzy numbers (a, b, c) with $a > 0$.

We note the fact that the operations "? " or "? " defined before (using $m * h$ or confidence intervals) do not necessarily lead to triangular numbers if one starts with triangular numbers. For instance, $\tilde{1} ?\ \tilde{1} = \begin{cases} \sqrt{t}, t \in [0,1]; \\ 0, \text{ otherwise} \end{cases}$ ($\tilde{1} :\ (1,1,1)$). This justifies somehow the operations defined above ("component-wise"), but the deviations for the variant given by "∗" for product and quotient are considerable. On the other hand, one obtains for usual real numbers (considered as fuzzy numbers) the usual operations. The problem of building an acceptable arithmetic for fuzzy number is still open.

4.2 Similarity relations and partitions

The role played by the notion of *relation* in the structure of mathematical concepts is well known. We review known results on the introduction of this notion in the framework of fuzzy set theory.

2.1 DEFINITION. Let X and Y be sets. We call a *fuzzy relation* between X and Y any fuzzy subset $r : X \times Y$? $[0,1]$ of the (usual) cartesian product $X \times Y$. If $X = Y$, we say that r is a *fuzzy relation on X*. Let $\mathsf{R}_f(X)$ be the set of all fuzzy relations on X.

The *diagonal* fuzzy relation on X is $\Delta : X \times X$? $[0,1]$, $\Delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$.

If $r : X \times Y$? $[0,1]$ is a fuzzy relation between X and Y, then $r^{-1} : Y \times X$? $[0,1]$ defined by $r^{-1}(y, x) = r(x, y)$ is called the *inverse* of r.

In the same manner as in the classical case, since the fuzzy relations are, in fact, fuzzy subsets, one may introduce the operations ? and n with fuzzy relations, as well as defining the inclusion between the fuzzy relations. Among the many possibilities of *composing* the fuzzy relations, we present the definition due to ZADEH:

Let X, Y, Z be sets and $r : X \times Y \to [0, 1]$, $x : Y \times Z \to [0, 1]$ fuzzy relations. The *composition of the fuzzy relations* r and x is the fuzzy relation $r \circ x : X \times Z \to [0, 1]$, defined by $r \circ x(x, z) = \sup_{y \in Y} \inf\{r(x, y), x(y, z)\}$.

For $r \in \mathsf{R}_f(X)$, we set $r^0 = \Delta$ and $r^{n+1} = r^n ? r$, $\forall n \in \mathbb{N}$.

2.2 PROPOSITION. *i) If* $r_1 : X \times Y \to [0, 1]$, $r_2 : Y \times Z \to [0, 1]$, $r_3 : Z \times U \to [0, 1]$ *are fuzzy relations, then* $(r_1 \circ r_2) \circ r_3 = r_1 \circ (r_2 \circ r_3)$.

ii) Let $r : Y \times Z \to [0, 1]$, r_1 *and* $r_2 : X \times Y \to [0, 1]$ *be fuzzy relations such that* $r_1 \subseteq r_2$. *Then* $r_1 \circ r \subseteq r_2 \circ r$.

iii) Let $r : Y \times Z \to [0, 1]$, r_1 *and* $r_2 : X \times Y \to [0, 1]$ *be fuzzy relations. Then* $(r_1 \cup r_2) \circ r = (r_2 \circ r) \cup (r_2 \circ r)$ *and* $(r_1 \cap r_2) \circ r \subseteq (r_2 \circ r) \cap (r_2 \circ r)$.

2.3 DEFINITION. Let r be a fuzzy relation on a fuzzy set (X, m).

- r is called *reflexive* if $r(x, x) = m(x)$, for any $x \in X$ ($r(x,x) = 1$ for an usual set);
- r is called *symmetric* if $r(x, y) = r(y, x)$, for any $(x, y) \in X \times X$;
- r is called *Z-transitive* if $r(x, z) \geq \sup_{y \in X} \min\{r(x, y), r(y, z)\}$, for any $x, z \in X$;

The fuzzy counterpart of the classical equivalence relation is the *similarity relation*.

2.4 DEFINITION. A relation $r : X \times X \to [0, 1]$ is called a *similarity relation* on X if it is reflexive, symmetric and Z-transitive.

2.5 PROPOSITION. *Let* $r : X \times X \to [0, 1]$ *be a similarity relation and* $x, y, z \in X$. *Then* $r(x, y) = r(y, z)$ *or* $r(x, z) = r(y, z)$ *or* $r(x, z) = r(x, y)$.

By using the level subsets, one obtains:

2.6 PROPOSITION. *The relation* $r : X \times X \to [0, 1]$ *is a similarity relation if and only if, for any* $a \in [0, 1]$, ${}_{r}(X \times X)_{a}$ *is an equivalence relation on X.*

2.7 PROPOSITION. *Let* $r : X \times X \to [0, 1]$ *be a fuzzy relation on X. The smallest similarity relation* r_s *with* $r \subseteq r_s$ *is* $r_s(x, y) = \sup \{(r?\ \Delta?\ r^{-1})^n(x, y) \mid n \in \mathbb{N}\}$.

The notion of equivalence class leads, in this setting, to the notion of similarity class. Let $r : X \times X \to [0, 1]$ be a similarity relation and $x \in X$. The *similarity class of representative x* is $r_x : X \to [0, 1]$, $r_x(y) = r(x, y)$, for any $y \in X$. Unlike the equivalence classes, the similarity classes are not necessarily disjoint (with respect to fuzzy intersection).

We point out some connections with the *fuzzy partitions*. Let X be a set and $J = \{1, 2, \ldots, n\}$. The symbols " · " , "? ", "? " denote the operations on F (X) defined at 2.1.6.

2.8 DEFINITION. The fuzzy sets $m_1, m_2, \ldots, m_n \in$ F (X) are called:

- *s-disjoint*, if, $\forall k \in J$, $(?_{i\in J\text{-}\{k\}} m_j)?\ m_k = \varnothing$;
- *w-disjoint*, if $?_{1 \le i \le n}\, m_j = \varnothing$;
- *i-disjoint*, if, $\forall r, s \in J$, $r \ne s$, $m_r \cap m_s = \varnothing$;
- *t-disjoint*, if, $\forall r, s \in J$, $r \ne s$, $m_r \cdot m_s = \varnothing$.

We say that the letters s, w, i, t are associated, respectively, to the operations "? ", "? ", "∩ ", "·".

2.9 REMARK. The above definitions can be extended in a natural manner to a *countable family* of fuzzy sets of F (X): $\forall a \in \{s, w, i, t\}$, $m_1, m_2, \ldots, m_n, \ldots \in$ F (X) are called *a-disjoint* if, for any $n \in \mathbb{N}$, $m_1, m_2, \ldots, m_n$ are a-disjoint.

2.10 REMARK. a) If $m_1 \cap m_2 = \varnothing$ then $m_1 ?\ m_2 = \varnothing$. The converse is not generally true. It is true if m_1 and m_2 are characteristic functions.

b) $m_1 ?\ m_2 = \varnothing \Leftrightarrow (m_1 ?\ m_2)(x) = m_1(x) + m_2(x)$, $\forall x \in X$.

c) Let $(A_i)_{i\in J}$ a family of n subsets of X and let c_i the characteristic function of A_i, $\forall i \in J$. Then c_i, $i \in J$, are s-disjoint if and only if, $\forall i, j \in J$, $i \ne j$ implies $c_i ?\ c_j = \varnothing$.

d) $m_1 \cap m_2 = \varnothing$ if and only if $m_1 \cdot m_2 = \varnothing$.

e) $m_1, m_2, \ldots, m_n$ are s-disjoint $\Rightarrow m_1, m_2, \ldots, m_n$ are w-disjoint.

2.11 PROPOSITION. We have:

a) $m_1, m_2, \ldots, m_n$ *are s-disjoint* $\Leftrightarrow \forall x \in X$, $m_1(x) + m_2(x) + \ldots + m_n(x) \le 1$;
b) $m_1, m_2, \ldots, m_n$ *are s-disjoint* $\Leftrightarrow \forall x \in X$, $\Sigma_{i\in J}\, m_i(x) = ?_{i\in J}\, m_i(x)$;
c) $m_1, m_2, \ldots, m_n$ *are w-disjoint* $\Leftrightarrow \forall x \in X$, $m'_1(x) + m'_2(x) + \ldots + m'_n(x) \le 1$;
d) $m_1, m_2, \ldots, m_n$ *are w-disjoint* $\Leftrightarrow \forall x \in X$, $m_1(x) + m_2(x) + \ldots + m_n(x) \le n - 1$.

Correspondingly, we obtain the notion of *s-partition* with $s \in \{s, w, i, t\}$.

2.12 DEFINITION. Let s be an element of $\{s, w, i, t\}$ and let ω be the associated operation. The family $\{m_i\}_{i\in J} \subseteq$ F (X) is called a *fuzzy s-partition* of $m \in$ F (X) if $m_1, m_2, \ldots, m_n$ are s-disjoint and $\omega_{i\in J}\ m_i = m$. Similarly, one can define the *countable partitions* of a fuzzy subset of X. When $m = c_A$, with A subset of X, the s-partition is called a *fuzzy s-partition* of A.

2.13 Remark. If $\{m_1, m_2, \ldots, m_n\}$ is an s-partition of m and $n \le m$, then $\{n \cdot m_1, n \cdot m_2, \ldots, n \cdot m_n\}$ is an s-partition for $n \cdot m$.

Let $r : X \times X \to [0,1]$ be a non-degenerate similarity relation (there exist $x, y \in X$, $x \neq y$, such that $r(x, y) = 1$). In the following we consider that X is a finite or countable set. For any $x \in X$ we denote m_x: $X \to [0,1]$ the function such that $m_x(y) = 1$ if $r(x, y) = 1$ and $m_x(y) = 0$ if $r(x, y) \neq 1$.

2.14 Proposition. *In the conditions above, if* $\exists z \in X$ *such that* $m_x(z) = m_y(y) = 1$, then $m_x = m_y$.

The relation on X, defined by $x \sim y$ if and only if $m_x = m_y$, is an equivalence relation on X. Let $K = X/\sim$ and denote by $[x]$ the class of x, $\forall x \in X$. Define $m_{[x]} = m_x$.

2.15 Proposition. *The set* $H = \{m_{[x]} \mid x \in X\}$ *is a fuzzy w-partition and a fuzzy i-partition of* X.

4.3 Connections between hyperstructures and fuzzy sets

The connections between algebraic hyperstructures and the fuzzy sets may take into account the following variants:

A. Let H be a nonempty set. One may replace (in the definition 2.2.1 of a hyperoperation on H) $\mathcal{P}^*(H)$ with $\mathcal{F}^*(H)$, where $\mathcal{F}^*(H) = \{m : H \to [0, 1] : \exists x \in H$ such that $m(x) \neq 0\}$ (the “family of nonempty fuzzy subsets of H”).

B. For a given hyperstructure, define a *fuzzy subhyperstructure* in an analogous manner to the one used to introduce the fuzzy subgroups.

C. Associating a hyperstructure to a fuzzy set (and conversely).

Concerning the variant **A** above, we have:

3.1 Definition. Let H be a nonempty set. An application $\bullet : H \times H\ ?\ \mathcal{F}^*(H)$ is called an *f-hyperoperation* on H.

For $a, b \in H$, $K \in \mathcal{P}^*(H)$, $m \in \mathcal{F}^*(H)$, we define:

$a ? b = \{x \in H \mid (a \bullet b)(x) \neq 0\}$, $a ? K = \bigcup_{k \in K} a ? k$, $K ? a = \bigcup_{k \in K} k ? a$;

$a ? b = \{x \in H \mid (a \bullet b)(x) = 1\}$, $a ? K = \bigcup_{k \in K} a ? k$, $K ? a = \bigcup_{k \in K} k ? a$;

$a \bullet K \in \mathcal{F}^*(H)$, $(a \bullet K)(x) = \sup\{(a \bullet k)(x) \mid k \in K\}$, $\forall x \in H$.

$K \bullet a \in \mathcal{F}^*(H)$, $(K \bullet a)(x) = \sup\{(k \bullet a)(x) \mid k \in K\}$, $\forall x \in H$.

$a \bullet m = a \bullet \mathrm{supp}(m)$; $m \bullet a = \mathrm{supp}(m) \bullet a$, where $\mathrm{supp}\, m := \{x \in H \mid m(x) \neq 0\}$.

We introduce some conditions related to *reproducibility*. We say that the f-hyperoperation "$\bullet$" on H satisfies the condition:

(R_1) if: $a \bullet H = c_H = H \bullet a$, $\forall a \in H$;

(R_2) if: $a ? H = H = H ? a$, $\forall a \in H$;

(R_3) if: $a ? H = H = H ? a$, $\forall a \in H$.

3.2 Definition. A nonempty set H endowed with an f-hyperoperation $\bullet$ is called an *f_i-hypergroup* ($i \in \{1, 2, 3\}$) if "$\bullet$" is associative ($a \bullet (b \bullet c) = (a \bullet b) \bullet c$, $\forall a, b, c \in H$) and satisfies the condition R_i.

3.3 PROPOSITION. *a)* $(H, \bullet)$ *is a* f_3*-hypergroup* $\Rightarrow$ $(H, \bullet)$ *is a* f_1*-hypergroup* $\Rightarrow$ $(H, \bullet)$ *is a* f_2*-hypergroup*.

b) For any $i \in \{1, 2, 3\}$, *if* $(H, \bullet)$ *is a* f_i*-hypergroup, then* $(H, ?\ )$ *is a hypergroup*.

c) If $(H, *)$ *is a hypergroup, then* $(H, \bullet)$ *is a* f_i*-hypergroup, for any* $i \in \{1, 2, 3\}$, *where* $\bullet : H \times H\ ?\ \ \mathrm{F}^{*}(H)$ *is given by* $(a\bullet b)(x) = \begin{cases} 1 & \text{if } x \in a * b \\ 0 & \text{otherwise} \end{cases}$.

The variant **C** above can be used in the following manner: if $m : A \to L$ is an L-fuzzy set, where $(L, \wedge,\vee)$ is a lattice, define the following *hyperoperation* on *A:*

(1) $$a*b = \{x \in A :\ m(a)\wedge m(b) \le m(x) \le m(a)\vee m(b)\},$$

where "$\le$" is the order relation on L.

3.4 PROPOSITION. *In the conditions above, for every* $a, b, c \in A$, *we have*:

i) $a \in a*b$;

ii) $a*b = b*a$;

iii) $a*(a*b) = a*b = (a*a)*b = (a*a)*(b*b) = (a*b)*b$.

3.5 PROPOSITION. *If* $m(L)$ *is a distributive sublattice in L (it is stable with respect to the operations* $\wedge$ *and* $\vee$ *and* $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$, *for any* $a, b, c \in m(L)$), *then*:

iv) $(a*b)*c = a*(b*c)$, *for every* $a, b, c \in A$.

From 3.4.*i*) it follows at once that $a*A = A*a$, for any a in A. Together with 3.4.*ii*) and 3.5.*iv*), this allows us to say that $(A, *)$ is a *commutative hypergroup* if $m(L)$ *is a distributive sublattice in L*. Moreover, 3.4.*iii*) shows that $(a*a)*(b*b) = a*b$; the set $a*b$ depends only on $a*a$ and $b*b$.

3.6 QUESTION. A natural problem arises: *characterize the lattices L (e.g. by means of identities) with the property that the hyperoperation induced on L (viewed as an L-fuzzy set by* $1_L : L \to L$*) as in (1) is associative.* The result 3.4.*iv*) says that the class of lattices with this property includes the distributive lattices. In the case $L = [0, 1]$ (or, more generally, a totally ordered set), the hypergroup obtained above is even a *join space*.

Suppose now that $m(L)$ is a sublattice which possesses a greatest element denoted 1 (that is, $x \le 1$ for any x in $m(L)$). We then have the additional properties:

3.7 PROPOSITION. *In the conditions above, there exists* $w \in A$, *such that:*

v) *For any* $a, b \in A$, *the condition* $a*w = b*w$ *implies* $a*a = b*b$;

vi) *For any* $a, b \in A$, *there exist* $m, M \in A$ *such that*

$$M*w = \bigcap\{x*w : x \in a*b\}$$

$$\text{and } \bigcap\{x*w : x*w \supseteq \{a,b\}\} = m*w.$$

Let us consider the reverse problem: *given a hyperstructure* $(H, *)$ *satisfying the properties i)-iii) and v)-vi) above, can one find a lattice L and a mapping* $m : H \to L$ *such that "*$*$*" is the hyperoperation induced by* m*, as in (1)?* In order to answer this, let H satisfy the properties above. Define a relation "~" on H by:

$$a\sim b \text{ iff } a*a = b*b.$$

One readily checks that this is an *equivalence relation* on A. Let L be the factor set $H/\sim$ (the set $\{\hat{a} : a \in H\}$, where $\hat{a} = \{x \hat{I} H : x\sim a\}$ is the equivalence class of a). Define a relation r on L by:

$$\text{for any } a, b \in L,\ \hat{a}\, r\, b \text{ iff } b*b \subseteq a*w.$$

The relation r is *well defined* (does not depend on the representatives a and b). This is in fact an *order relation on L* and the ordered set (L, r) is a *lattice.* Define now the application $m : H \to L$ as the canonical projection: $m(a) = \hat{a}$, for any $a \in H$; define the hyperoperation "$*$" in H as in (1).

3.8 PROPOSITION. *In the conditions above, for any a and b in H,*

$$a*b = \{x \in A : m(a) \wedge m(b) \le m(x) \le m(a) \vee m(b)\}.$$

Acknowledgements

The authors wish to thank Professor H. N. Teodorescu for his support and competent advice given during the preparation of this paper.

REFERENCES

[1] Alkhamees, Y., Mordeson, J.: *Fuzzy principal ideals and fuzzy simple extensions*, Fuzzy Sets and Systems, 3 (1995), 911-919.
[2] Alkhamees, Y., Mordeson, J.: *Reduced fields, primitive and fuzzy Galois theory*, J. Fuzzy Math., vol. 8, No. 1 (2000), 157-173.
[3] Corsini, P. : *Prolegomena on Hypergroup Theory*, Aviani Ed., 1993.
[4] Corsini, P., Tofan, I. : *On fuzzy hypergroups*, PUMA, vol. 8, nr. 1, 1997, 29-37.
[5] Erceq, M. A.: *Functions, Equivalence Relations, Quotient Spaces and Subsets in Fuzzy Set Theory*, Fuzzy Sets and Systems **3** (1980), 75-92.
[6] Kerre, E. E., *Introduction to the Basic principles of the Set Theory and some of its Applications,* Comm. & Cognition, Gent, Belgium, 1991.
[7] Kumar, R., *Fuzzy Algebra* I, Univ. Delhi Publ. Division, 1993.
[8] Liu, Wang-Jiu: *Fuzzy Invariant Subgroups and Fuzzy Ideals*, F.S.S., vol. **8** (1982), 133-139.
[9] Liu, Wang-Jiu: *Operations on Fuzzy Ideals*, F.S.S., vol. **8** (1983), 31-41.
[10] Maturo, A. Tofan, I, *Similitude relations and applications in Architecture*, communication, SIGEF Conference, Napoli 2001.
[11] Mordeson, J.N.: *Fuzzy Galois theory*, J. Fuzzy Math.(1993), 659-671.
[12] Mukherjee, T.K., Sen, M.K. : *On Fuzzy Ideals of a Ring* (1), F.S.S vol. **21** (1987), 99-104.
[13] Nistor, S., Tofan, I., Volf, A.C., *Fuzzy algebraic structures*, Fuzzy Systems &A.I., Acad. Româna, vol 5, nr. 1 (1996), 5-15.
[14] Ovchinnikov, S. V., *Structure of Fuzzy Binary Relations*, Fuzzy Sets and Systems **6** (1981), 169-197.
[15] Teodorescu, H.N., *Binary-like codes for trapezoidal fuzzy numbers*, Vol. Proc. 3rd IFSA Congress, Washington, 1989, 548-551.
[16] Teodorescu, H.N., *Congruences of fuzzy numbers*, Proc. Int. Poznan Seminar on Fuzzy Sets and Interval Analysis, 1988.
[17] Teodorescu, H.N., in cooperation with S. Nistor, I. Tofan, *Fuzzy Sets, part 2: Fuzzy arithmetic,* R.R., Ecole Pol. Féd. Lausanne 1994.
[18] Tofan, I., *Toposes of totally fuzzy sets,* BUSEFAL **58** (1994), 60-65.
[19] Tofan, I., Volf, A.C., *Fuzzy rings of quotients*, Italian Journal of Pure and Applied Mathematics no. **8** (2000), 83-90.
[20] Tofan, I., Volf, A.C., *On some conections between hyperstructures and fuzzy sets*, Italian Journal of Pure and Applied Mathematics, no. **7** (2000), 63–68.
[21] Tofan, I., Volf, A.C., *Some remarks on fuzzy relations*, Fuzzy Systems &A.I., Acad. Româna, vol 4, nr. 2 (1995), 1-4.
[22] Zadeh, L. A. : Fuzzy sets, *Inform. and Control*, vol. **8** (1965), 338-353.
[23] Zahedi, M.M., Bolurian, M., Hasankhani, A., *On polygroups and fuzzy subpolygroups*, J. of Fuzzy Math., vol 2, nr. 4, 1994, 1-15.